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ON THE STEIN PROPERTY OF RADEMACHER SEQUENCES

BY

KRZYSZTOF OLESZKIEWICZ (WARSZAWA)

Abstract. We prove that for a Rademacher sequence (r_i) and any sequence of real numbers (a_i) the inequality

$$P\left(\left|\sum_{i=1}^{n} a_i r_i\right| \geqslant \sqrt{\sum_{i=1}^{n} a_i^2}\right) \geqslant \frac{1}{10}$$

holds true.

Let (r_i) be a *Rademacher sequence* (i.e., a sequence of independent symmetric random variables taking on values 1 and -1). For each natural number n let us denote by C_n the greatest real number such that for any real numbers $a_1, a_2, \ldots, a_n$ the inequality

$$P\left(\sum_{i=1}^{n} a_i r_i \geqslant \sqrt{\sum_{i=1}^{n} a_i^2}\right) \geqslant C_n$$

holds true. We put $C = \inf_n C_n$.

Burkholder [1] proved that $C > 0$, however an estimate for C which can be derived from his proof is rather far away from being optimal. Hitczenko and Kwapień [2] proved that $C > 1/8e^4$ and stated without proof that $C > \frac{1}{40}$. The constant C is useful in Kolmogorov-type lower estimates for tails of sums of independent random variables and in some other problems (cf. [1] and [2]). Here we give a simple and elementary proof that $C > \frac{1}{20}$.

THEOREM 1. *For $n > 1$ and real numbers $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n > 0$ let*

$$S = \sum_{i=1}^{n} a_i r_i \quad \text{and} \quad \sigma = \sqrt{\sum_{i=1}^{n} a_i^2}.$$

Then $P(S > \sigma) > \frac{1}{20}$.

LEMMA 1. *Under the assumptions of Theorem 1 we have*

$$P\left(S > \frac{\sigma}{\sqrt{3}}\right) > \frac{1}{10}.$$

Proof of Lemma 1. By the homogeneity of the formula we can assume that $\sigma = \pi/2$. Since

$$(1)\qquad 0 \leqslant \cos x \leqslant \exp\{-x^2/2\} \quad \text{for } |x| \leqslant \pi/2,$$

we obtain

$$\exp\{-\pi^2/8\} = \exp\{-\sigma^2/2\} = \prod_{i=1}^{n} \exp\{-a_i^2/2\} \geqslant \prod_{i=1}^{n} \cos a_i = \mathrm{E}\cos S$$
$$\geqslant \cos\frac{\pi}{2\sqrt{3}}\, P\left(|S| \leqslant \frac{\pi}{2\sqrt{3}}\right) - P\left(|S| > \frac{\pi}{2\sqrt{3}}\right)$$
$$= \cos\frac{\pi}{2\sqrt{3}} - 2\left(1+\cos\frac{\pi}{2\sqrt{3}}\right) P\left(S > \frac{\sigma}{\sqrt{3}}\right).$$

Hence

$$P\left(S > \frac{\sigma}{\sqrt{3}}\right) \geqslant \frac{\cos(\pi/2\sqrt{3}) - \exp\{-\pi^2/8\}}{2(1+\cos(\pi/2\sqrt{3}))} > \frac{1}{10},$$

and this proves Lemma 1.

Proof of Theorem 1. We can assume that $\sigma = \pi/2$. Let us consider two cases.

Case 1. $a_1 > \pi/4$. Let

$$S' = \sum_{i=2}^{n} a_i r_i \quad \text{and} \quad \sigma' = \sqrt{\sum_{i=2}^{n} a_i^2}\,.$$

We obtain $2a_i > \sigma$ so that

$$\sigma'^2 = \sigma^2 - a_1^2 = 3(\sigma - a_1)^2 + 2(2a_1 - \sigma)(\sigma - a_1) \geqslant 3(\sigma - a_1)^2,$$

and therefore $\sigma - a_1 \leqslant \sigma'/\sqrt{3}$. Hence

$$P(S > \sigma) \geqslant \frac{1}{2} P(S' > \sigma - a_1) \geqslant \frac{1}{2} P\left(S' > \frac{\sigma'}{\sqrt{3}}\right) > \frac{1}{2}\,\frac{1}{10} = \frac{1}{20},$$

according to Lemma 1.

Case 2. $a_1 \leqslant \pi/4$. Then

$$P(S > \sigma) = \tfrac{1}{2} P(|S| > \pi/2) \geqslant \tfrac{1}{2} P(\cos S < 0) = \tfrac{1}{4}(2P(\cos S < 0) + 0P(\cos S \geqslant 0))$$
$$\geqslant \tfrac{1}{4}\mathrm{E}(\cos^2 S - \cos S) = \tfrac{1}{8} + \tfrac{1}{8}\mathrm{E}\cos 2S - \tfrac{1}{4}\mathrm{E}\cos S$$

$$= \tfrac{1}{8}+\tfrac{1}{8}\prod_{i=1}^{n}\cos 2a_i-\tfrac{1}{4}\prod_{i=1}^{n}\cos a_i \geqslant \tfrac{1}{8}-\tfrac{1}{4}\prod_{i=1}^{n}\cos a_i$$

$$\geqslant \tfrac{1}{8}-\tfrac{1}{4}\prod_{i=1}^{n}\exp\{-a_i^2/2\} = \tfrac{1}{8}-\tfrac{1}{4}\exp\{-\sigma^2/2\} = \tfrac{1}{8}-\tfrac{1}{4}\exp\{-\pi^2/8\} > \tfrac{1}{20}.$$

Again we used the inequality (1). So the proof of Theorem 1 is completed.

An example of $n = 4$ and $a_1 = a_2 = a_3 = a_4 \neq 0$ shows that the constant $\frac{1}{20}$ in Theorem 1 cannot be replaced by any number greater than $\frac{1}{16}$.

A similar example of $n = 6$ and $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 \neq 0$ shows that $C \leqslant \frac{7}{64}$. It seems probable that, in fact, $C = \frac{7}{64}$, however we are only able to prove the following

THEOREM 2. $C_6 = \frac{7}{64}$.

Proof of Theorem 2. We can assume that $a_1 \geqslant a_2 \geqslant a_3 \geqslant a_4 \geqslant a_5 \geqslant a_6 \geqslant 0$. We will say that a combination of signs $(s_1, s_2, s_3, s_4, s_5, s_6) \in \{-, +\}^6$ is *good* if

$$\sum_{i=1}^{6} a_i s_i \geqslant \sqrt{\sum_{i=1}^{6} a_i^2}\,.$$

We have to show that at least seven different combinations of signs are good.

Since

$$(a_1+a_2+a_3-a_4+a_5+a_6)^2$$
$$\geqslant \sum_{i=1}^{6} a_i^2+2(a_1-a_4)(a_3+a_5+a_6)+2a_1(a_2-a_4)+2a_2(a_3-a_4) \geqslant \sum_{i=1}^{6} a_i^2,$$

the combination $(+, +, +, -, +, +)$ is good, and therefore $(+, +, +, +, -, +)$, $(+, +, +, +, +, -)$ and $(+, +, +, +, +, +)$ are also good. We need three other combinations. One can easily check that

$$(a_1+a_2-a_3+a_4+a_5+a_6)^2+(a_1+a_2+a_3-a_4-a_5+a_6)^2 \geqslant 2\sum_{i=1}^{6} a_i^2.$$

Hence, if $(+, +, +, -, -, +)$ is good, then $(+, +, +, -, +, -)$ and $(+, +, +, +, -, -)$ are good as well (and we are done), so that we can assume that $(+, +, -, +, +, +)$ is good. Now we need only two more combinations of signs. It is also easy to prove that

$$(a_1-a_2+a_3+a_4+a_5+a_6)^2+(a_1+a_2+a_3-a_4+a_5-a_6)^2 \geqslant 2\sum_{i=1}^{6} a_i^2.$$

Similarly, if $(+, +, +, -, +, -)$ is good, then also $(+, +, +, +, -, -)$ is good, so that we can assume that $(+, -, +, +, +, +)$ is good and we need only one more combination of signs. An easy argument shows that

$(-, +, +, +, +, +)$ is such a combination if $a_1 \leqslant a_5+a_6$ and $(+, +, +, +, -, -)$ does if $a_1 > a_5+a_6$. The proof is completed.

Remark. It was proved by Holzman and Kleitman [3] that

$$P\left(\left|\sum_{i=1}^{n} a_i r_i\right| > \sqrt{\sum_{i=1}^{n} a_i^2}\right) \leqslant \frac{5}{8}.$$

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Institute of Mathematics
Warsaw University
ul. Banacha 2, 02-097 Warszawa, Poland
e-mail: koles@mimuw.edu.pl

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